

## ON AN INDEX THEOREM BY BISMUT

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ABSTRACT. In this paper we give a proof of an index theorem by Bismut. As a consequence we obtain another proof of the Grothendieck–Riemann–Roch theorem in differential cohomology.

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## 1. INTRODUCTION

The differential Grothendieck–Riemann–Roch theorem [7, Theorem 6.19], [11, Corollary 8.26], [12, Theorem 1] (abbreviated as dGRR) is a lift of the classical Grothendieck–Riemann–Roch theorem to differential cohomology. It states that for a proper submersion  $\pi : X \rightarrow B$  with closed  $\text{spin}^c$  fibers of even relative dimension, the following diagram commutes.

$$\begin{array}{ccc}
 \widehat{K}(X) & \xrightarrow{\widehat{\text{ch}}} & \widehat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\
 \text{ind}^{\text{an}} \downarrow & & \downarrow \widehat{\int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) * (\cdot)} \\
 \widehat{K}(B) & \xrightarrow[\widehat{\text{ch}}]{} & \widehat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q})
 \end{array} \quad (1)$$

Here  $\widehat{K}$  is differential  $K$ -theory [8, 17, 11] and  $\widehat{H}$  is Cheeger–Simons differential characters [9, 1].

In [12] the proof of the dGRR is to reduce it to an index theorem by Bismut [4, Theorem 1.15]:

$$\widehat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\widetilde{\eta}(\mathcal{E})) = \widehat{\int_{X/B} \text{Todd}(T^V X, \widehat{\nabla}^{T^V X}) * \widehat{\text{ch}}(E, \nabla^E)}. \quad (2)$$

One can regard (2) as a lift of the local family index theorem [5] to differential characters. Bismut's proof of (2) involves certain adiabatic limits arguments given in [5, 10, 16] and an Atiyah–Patodi–Singer index theorem in differential characters [9, Theorem 9.2]. In this paper we give a proof of (2), which is inspired by [1] and does not make use of the above results.

Section 2 contains the background material needed in this paper. Section 3 contains the main result of this paper.

## ACKNOWLEDGEMENT

The author would like to thank Thomas Schick for pointing out a mistake in an earlier version of the paper.

## 2. BACKGROUND MATERIALS

**2.1. Cheeger–Simons differential characters.** We recall some basic properties of differential characters, and refer to [9, 1] for the details.

Let  $X$  be a smooth manifold and  $k \geq 1$ , and  $A$  a proper subring of  $\mathbb{R}$ . A degree  $k$  differential character  $f$  with coefficients in  $\mathbb{R}/A$  is a group homomorphism  $f : Z_{k-1}(X) \rightarrow \mathbb{R}/A$  with a fixed  $\omega_f \in \Omega^k(X)$  such that for all  $c \in C_k(X)$ ,  $f(\partial c) = \int_c \omega_f \pmod A$ . The abelian group of degree  $k$  differential characters is denoted by  $\hat{H}^k(X; \mathbb{R}/A)$ . Denote by  $\Omega_A^k(X)$  the group of closed  $k$ -forms with periods in  $A$ . It is easy to see that  $\omega_f \in \Omega_A^k(X)$  and is uniquely determined by  $f \in \hat{H}^k(X; \mathbb{R}/A)$ . The map  $\delta_1 : \hat{H}^k(X; \mathbb{R}/A) \rightarrow \Omega_A^k(X)$  by  $\delta_1(f) = \omega_f$ . The map  $i_2 : \frac{\Omega^{k-1}(X)}{\Omega_A^{k-1}(X)} \rightarrow \hat{H}^k(X; \mathbb{R}/A)$ , defined by  $i_2(\alpha)(z) = \int_z \alpha \pmod A$ , is injective. In the following diagram, every square and triangle commutes, and the two diagonal sequences are exact.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & & \nearrow & \\
 & & H^{k-1}(X; \mathbb{R}/A) & \xrightarrow{-B} & H^k(X; A) & \nearrow 0 \\
 & \nearrow \alpha & & \searrow \delta_2 & & \\
 H^{k-1}(X; \mathbb{R}) & & \hat{H}^k(X; \mathbb{R}/A) & & H^k(X; \mathbb{R}) \\
 & \searrow \beta & \nearrow i_1 & \searrow \delta_1 & \nearrow s \\
 & & \frac{\Omega^{k-1}(X)}{\Omega_A^{k-1}(X)} & \xrightarrow{d} & \Omega_A^k(X) \\
 & \nearrow & & & \searrow 0 \\
 0 & & & & 
 \end{array} \tag{3}$$

The maps  $\delta_1$  and  $\delta_2$  are called the curvature and the characteristic class in literatures respectively.

There is a unique ring structure for  $\widehat{H}^*(X; \mathbb{R}/A)$  [1, Corollary 32], denoted by  $*$ . For a fiber bundle  $\pi : X \rightarrow B$  with closed oriented fibers, the “integration along the fiber”, denoted by  $\widehat{\int_{X/B}}$ , exists and is unique [1, Theorem 39].

**2.2. Index bundle and Bismut–Cheeger eta form.** In this subsection we recall the construction of the index bundle and of the Bismut–Cheeger eta form. We refer to [2, 15] for the details.

Let  $E \rightarrow X$  be a complex vector bundle with a Hermitian metric  $h$  and  $\nabla$  a unitary connection on  $E \rightarrow X$ . Let  $\pi : X \rightarrow B$  be a proper submersion of even relative dimension  $n$ , and  $T^V X \rightarrow X$  the vertical tangent bundle which is assumed to have a metric  $g^{T^V X}$ . A given horizontal distribution  $T^H X \rightarrow X$  and a Riemannian metric  $g^{TB}$  on  $B$  determine a metric on  $TX \rightarrow X$  by  $g^{TX} := g^{T^V X} \oplus \pi^* g^{TB}$ . If  $\nabla^{TX}$  is the corresponding Levi-Civita connection on  $TX \rightarrow X$ , then  $\nabla^{T^V X} := P \circ \nabla^{TX} \circ P$  is a connection on  $T^V X \rightarrow X$ , where  $P : TX \rightarrow T^V X$  is the orthogonal projection. Assume the vertical bundle  $T^V X \rightarrow X$  has a  $\text{spin}^c$ -structure; i.e., the principal  $\text{SO}(n)$ -bundle  $\text{SO}(T^V X) \rightarrow X$  admits a  $\text{spin}^c(n)$  reduction. Denote by  $S^c(T^V X) \rightarrow X$  the spinor bundle, which is a complex vector bundle, associated to the  $\text{spin}^c$  structure of  $T^V X \rightarrow X$ . Note that  $S^c(T^V X) \cong S(T^V X) \otimes L^{\frac{1}{2}}(X)$ , where  $S(T^V X)$  is the spinor bundle associated to the locally defined spin structure of  $T^V X \rightarrow X$ , and  $L^{\frac{1}{2}}(X)$  is a locally defined Hermitian line bundle (see [15, (D.19)]) Note that their tensor product is globally defined. Write  $\nabla^{T^V X}$  for both the Levi-Civita connection on  $T^V X \rightarrow X$  and also its lift to  $S(T^V X)$ . Choose a unitary connection  $\nabla^{L^V X}$  on  $L^{\frac{1}{2}}(X)$ . Define a connection  $\widehat{\nabla}^{T^V X}$  on  $S^c(T^V X) \rightarrow X$  by  $\widehat{\nabla}^{T^V X} := \nabla^{T^V X} \otimes \nabla^{L^V X}$ . The Todd form  $\text{Todd}(\widehat{\nabla}^{T^V X})$  of  $S^c(T^V X) \rightarrow X$  is defined by

$$\text{Todd}(\widehat{\nabla}^{T^V X}) := \widehat{A}(\nabla^{T^V X}) \wedge e^{\frac{1}{2}c_1(\nabla^{L^V X})}.$$

The Bismut–Cheeger eta form  $\widetilde{\eta}(\mathcal{E}) \in \frac{\Omega^{\text{odd}}(B)}{\text{Im}(d)}$  associated to  $\mathcal{E} := (E, h, \nabla)$  is defined as follows. Consider the infinite-rank superbundle  $\pi_* E \rightarrow B$ , where the fibers at each  $b \in B$  is given by

$$(\pi_* E)_b := \Gamma(X_b, (S^c(T^V X) \otimes E)|_{X_b}).$$

Recall that  $\pi_* E \rightarrow B$  admits an induced Hermitian metric and a connection  $\nabla^{\pi_* E}$  compatible with the metric [2, §9.2, Proposition 9.13]. For each  $b \in B$ , the canonically constructed Dirac operator

$$D_b^E : \Gamma(X_b, (S^c(T^V X) \otimes E)|_{X_b}) \rightarrow \Gamma(X_b, (S^c(T^V X) \otimes E)|_{X_b})$$

gives a family of Dirac operators, denoted by  $D^E : \Gamma(X, S^c(T^V X) \otimes E) \rightarrow \Gamma(X, S^c(T^V X) \otimes E)$ . Assume the family of kernels  $\ker(D_b^E)$  has locally constant dimension, i.e.,  $\ker(D^E) \rightarrow B$  is a finite-rank Hermitian superbundle. Let  $P : \pi_* E \rightarrow \ker(D^E)$  be the orthogonal projection,  $h^{\ker(D^E)}$  be the Hermitian metric on  $\ker(D^E) \rightarrow B$  induced by  $P$ , and  $\nabla^{\ker(D^E)} := P \circ \nabla^{\pi_* E} \circ P$  be the connection on  $\ker(D^E) \rightarrow B$  compatible to  $h^{\ker(D^E)}$ .

The scaled Bismut superconnection  $\mathbb{A}_t : \Omega(B, \pi_* E) \rightarrow \Omega(B, \pi_* E)$  [3, Definition 3.2] (see also [2, Proposition 10.15] and [10, (1.4)]), is defined by

$$\mathbb{A}_t := \sqrt{t} D^E + \nabla^{\pi_* E} - \frac{c(T)}{4\sqrt{t}},$$

where  $c(T)$  is the Clifford multiplication by the curvature 2-form of the fiber bundle. The Bismut–Cheeger eta form  $\tilde{\eta}(\mathcal{E})$  [5, (2.26)] (see also [10] and [2, Theorem 10.32]) is defined by

$$\tilde{\eta}(\mathcal{E}) := \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \text{Str} \left( \frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) dt.$$

The local family index theorem states that

$$d\tilde{\eta}(\mathcal{E}) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^{\ker(D^E)}). \quad (4)$$

Let  $f : \tilde{B} \rightarrow B$  be a smooth map. In [7, §2.3.2] the pullback of the above geometric data is studied, and, in particular, the Bunke eta form ([6, Definition 2.2.16]) is shown to respect pullback. Since the Bismut–Cheeger eta form is a special case of the Bunke eta form, we have

$$\tilde{\eta}(f^* \mathcal{E}) = f^* \tilde{\eta}(\mathcal{E}). \quad (5)$$

One can also prove (5) directly as in [7, §2.3.2].

### 3. MAIN RESULT

In this section we prove the main result in this paper. We employ the setup and assumptions made in Section 2.2. For write  $\mathcal{E}$  for  $(E, h, \nabla)$ , where  $E \rightarrow X$  is a Hermitian bundle with a Hermitian metric  $h$  and  $\nabla$  a unitary connection on  $E \rightarrow X$ .

As in [12] it suffices to prove (2) in the special case where the family of kernels of the Dirac operators has constant dimension, i.e.,  $\ker(D^E) \rightarrow B$  is a superbundle. The general case of (2) follows from a standard perturbation argument as in [11, §7] and its proof is essentially the same as the special case.

**Proposition 1.** For any  $\mathcal{E}$ , the differential character

$$\widehat{\int_{X/B} \text{Todd}(T^V X, \widehat{\nabla}^{T^V X}) * \widehat{\text{ch}}(E, \nabla) - \widehat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)})} \quad (6)$$

is uniquely characterized by the conditions that it is natural and its curvature is given by

$$\int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{ch}(\nabla) - \text{ch}(\nabla^{\ker(D^E)}). \quad (7)$$

*Proof.* The differential character in (6) obviously satisfies the conditions. The uniqueness of (6) come from [13, Proposition 3.1]. However, [13, Proposition 3.1] is only correct under a restrictive class of Lie groups<sup>1</sup>  $G$ , namely, it is valid if the cohomology of  $BG$  is torsion free. Example of such  $G$  includes the stable general linear group  $\text{GL}(\mathbb{C})$ , and therefore the stable unitary group  $U$ . Thus [13, Proposition 3.1] is true for all differential characteristic classes of complex vector bundles of even degree, and therefore it can still be applied to our case.  $\square$

Bismut's theorem follows from the observation that the differential character  $i_2(\tilde{\eta}(\mathcal{E})) \in \widehat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q})$  satisfies the conditions stated in Proposition 1: The naturality of  $i_2(\tilde{\eta}(\mathcal{E}))$  follows from (5). The curvature of  $i_2(\tilde{\eta}(\mathcal{E}))$  is given by (7) is a consequence of the commutativity of the lower triangle of (3) and the local family index theorem (4).

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<sup>1</sup>The author would like to thank Thomas Schick for bringing up this point.

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